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Explicit Examples of Conformal Invariance

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We study examples where conformal invariance implies rational critical indices, triviality of the underlying quantum field theory, and emergence of hypergeometric functions as solutions of the field equations.

KEY WORDS: rational critical indices; triviality; conformal symmetry.

Conformal symmetry, although present in very few realistic systems that are seen around in its exact form, has proved to be a very important symmetry in physics. The pure Maxwell system was shown to be conformally invariant at the start of the twentieth century (Bateman, 1910; Cunningham, 1909). The presence of a mass term, say when an electron couples to this system, ruins this symmetry, though. Scaling arguments, of late 1960s, brought this symmetry back into stage. Wilson (1969) laid the theoretical grounds whereas experimentally validated Bjorken scaling laws (Bjorken, 1969) showed that these ideas have applications in down-to-earth physics. Ever since then conformal symmetry, however badly broken it is in real world, makes its appearance in different branches of physics.

In this note we want to give some examples, obtained from Lagrangian field theory, for results derived by using more formal methods. These examples, when properly reinterpreted, serve as illustrations for these phenomena. All these phenomena may be found scattered in the literature. We think it is still worthwhile to mention all these related phenomena, and the different signatures of conformal symmetry in one paper.

We first want to focus on the relationship between rational critical indices and the presence of conformal symmetry in a field theoretical model, a problem which has been studied through several decades. A classical paper on this relation is the one written by Friedan–Qui–Shenker (FQS) (Friedan *et al.*, 1984) where it is shown that the presence of conformal symmetry in statistical mechanical models forces the critical indices for phase transitions to be rational numbers. The conformal

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symmetry was put to the model through the commutation relations obeyed by the generators, the so-called Virasoro algebra and the calculation was dependent on these commutations relations obeyed by the algebra. The models studied did not have explicit lagrangians. The correspondence with lagrangian models was through matching dimensions of known exactly solvable models. Here unitarity was the forcing constraint resulting in the rational values for the critical indices. When c, c being the coefficient of the central term in the Virasoro algebra, is greater than unity, we get field theoretical models instead of the statistical mechanical ones. In this case the unitarity condition does not impose new constraints; so, one gets a continuous set of solutions.

Calculations made by Belavin *et al.* (1984), before the FQS paper appeared, showed the occurrence of rational indices explicitly in their conformal invariant models. The analysis, again, did not start with a lagrangian system. Both of these papers were working in two dimensional Euclidean spaces.

It is well known that the conformal group is much richer in two dimensions compared to higher dimensions. In two dimensions, it is infinite dimensional. When the space has a Euclidean metric, any function that obeys the Cauchy–Riemann equations is locally conformal invariant. If we use the definition of conformal symmetry as used in general relativity, being the transformation that keeps the light-cone invariant, we see that any metric in two dimensions can be put to this form easily. We can find a transformation that sends our former metric to the same metric times a function. This property is not true for higher dimensional metrics, though.

A stricker definition of conformal symmetry will be the one defined using group theory. This definition coincides with the former one if the space is higher than two dimensional. Here we take the conformal group as $SO(d, 2)$, where d is the dimension of space–time. The transformation is expressed as the full Poincaré group, plus scaling transformations where x^μ goes to λx^μ, here λ is a constant, and special conformal transformations, where

$$x^\mu \to \frac{x^\mu - b^\mu x^2}{1 - 2bx + b^2 x^2}. \tag{1}$$

The group has

$$\frac{(d+2)(d+1)}{2} \tag{2}$$

parameters where $x^2 = x_\mu x^\mu$, $bx = b_\mu x^\mu$. The special conformal transformations are generated by

$$R_\mu = P_\mu + I P_\mu I \tag{3}$$

where P_μ is the generator of translations and I, the inversion, takes x to $\frac{-x}{x^2}$. The inversion is not a part of the connected part of the conformal group, except for free theories.

It will be of interest if the relationship between the rationality of the critical indices and the presence of conformal symmetry persists in higher dimensions as well. Schroer and Swieca studied the presence of global conformal symmetry in field theory in 1974 (Schroer *et al.*, 1975; Schroer and Swieca, 1974). Using their result, we see that the conformal blocks of Belavin–Polyakov–Zamolodchikov are not local fields, but local fields can be constructed out of them by summing over blocks (Schroer, 2000a). It was shown earlier (Hortaçsu *et al.*, 1972), that global conformal invariance is absent in a local field theoretical model unless the two-point function can be written as $(x - x')^{-2n}$ where n is a positive integer. In this paper two exactly solvable interacting models, as well as free fields were studied. The essence of the argument is the fact that R_μ generates a kind of conformal "time" rotations (Schroer, 2000a), and maps spacelike distances into timelike distances. Unless the two point function has support only on the light-cone, which is the case only when n is an integer, the special conformal transformation violates causality and the Wightman functions, therefore the whole theory is not invariant under conformal transformations. When n is an integer, we get a theory which is unitarily equivalent to a theory which is made out of product of free fields (Wightman, 1967).

Back in 1970s, Gürsey and Orfanidis (1973) gave all the representations of the group SO(2, 2), the conformal group in two dimensions in the stricker sense. They showed that global conformal invariance exists only when one uses the analytic series out of all possible ones. The analytic series for SO(2, 2) were related to *nontrivial interacting field theories* with two-point functions with integer powers, giving explicit examples of the presence of global conformal invariance in two models (Hortaçsu, 1973a), the Thirring and the derivative coupling models. It was shown that the representations of SO(2, 2) allowing such a symmetry correspond to discrete values of the coupling constant and spin, those values that will give two point functions with integer powers. The Klaiber solution (Klaiber, 1968) to the Thirring model uses two constants that are functions of the coupling constant g and spin s. Since the model is in two dimensions, spin does not have a physical meaning and, in principle, can take any value. When spin is equal to one half, as is the case for any true *spinor*, it is not possible to satisfy conformal invariance. Only for anomalous values of the spin value, conformal invariance can exist for certain discrete values of the coupling constant.

Here what is meant by *nontrivial field theories* is just a theory whose coupling constant is not zero. The catch is on deciding whether a Lagrangian theory which exhibits a Feynman series expansion is *actually nontrivial*. We know from the example of the ϕ^4 theory (Baker and Kincaid, 1979, 1981) that the presence of an interaction term in the Lagrangian form of the theory does not necessary mean that the theory is not *trivial*.

It should be recalled that in recent literature in Minkowski space, it is shown that conformal invariance may result in "trivial" theories (Schroer, 2001). Schroer has shown recently (Schroer, 2000b) that conformal invariance, realized in

Minkowski space, does not allow a particle interpretation unless one has a free field theory. One is led to a trivial theory by the vanishing of the LSZ limit (Pohlmeyer, (1969); Bucholz and Fredenhagen, 1977). Schroer also comments on (Schroer, 2000b) how the presence of conformal symmetry maps the long and short distances behavior into each other, resulting in the coalescing of these two points. All multiparticle thresholds collapse on top of each other resulting in anomalous dimensions. In the presence of anomalous dimensions, the LSZ limit vanishes, resulting in the loss of particle interpretation. These facts necessitate further concrete calculations to check whether *true nonfree behavior* is present using models which are conformal invariant.

There were attempts to carry out calculations for the electron–positron annihilation and electroproduction processes, using Thirring model, which exhibits conformal invariance, as the fundamental model describing these interactions (Hortaçsu, 1973b). The results were inconclusive, though. In the former case the cross-section was found to be proportional to a power of the transferred momenta. In the latter case the presence of generalized hypergeometric functions (Appell and Kampé de Fériet, 1926; Humbert, 1920–21) made a clear interpretation not possible. Thus we could not check whether these processes, based on this conformal invariant model, gave results different from the free field case.

Calculations made a little later, however, showed explicitly how conformal symmetry leads to a trivial theory. There were attempts to regularize conformal invariant models in $d = 4$ by unconventional models (Akdeniz *et al.*, 1982, 1983; Arik *et al.*, 1985). These models were shown to give rise to trivial theories (Arik and Hortaçsu, 1983; Hortaçsu, 1994) in the sense that physical processes, calculated using such models, gave the free field result. In this reference, two physical processes, electron–positron annihilation and the quark–electron structure functions were calculated using the $1/N$ expansion. Calculations were carried up to three loops, consistent with the $1/N$ expansion. The end result was exactly the one given by the free quark model. In the annihilation cross-section, the one loop is the free quark result. The higher loop contributions cancelled one another, ending up with the free quark result. In the latter case, only the lowest order tree diagram remained, all the diagrams with loops cancelling each other. The model scaled exactly in both of the processes studied. The logarithmic corrections of QCD were lacking. In this sense this result confirms Schroer's ideas about the triviality of field theoretical models possessing conformal symmetry. The presence of conformal symmetry in the model results in a free field theory, at best a generalized free field theory which is formed by the powers of free fields (Wightman, 1967).

Schroer (2000d; Rehren, 2000) thinks that, if one starts with AdS space, instead of the Minkowski, there may be a way out. The zero component of the generator of special conformal transformations, R^0, may act as the Hamiltonian in the new space. At this point we want to remark on examples using models based on the AdS spaces.

Recently there were papers on models where critical behavior was studied for the emergence of black holes by the presence of scalar fields. For amplitudes of scalar fields which are less than a critical value, no black hole exists. As the amplitude of the scalar field increases, there is a certain value beyond which we find a black hole which swallows the scalar field. Such models show what is called Choptuik scaling (Choptuik, 1993). During last years the BTZ black hole (Bãnados *et al.*, 1992,1993) and the closely related AdS models were studied in the presence of a scalar field. Two sets of results exist. When one takes a theory built around either the BTZ black hole (Birmingham, 2001; Birmingham *et al.*, 2001) or the AdS solution (Kim and Oh, 2001), one gets a fraction, actually just 1/2 for the critical index for black hole formation. We know that the $d = 3$ AdS solution is dual to a conformal invariant model in $d = 2$ [Maldecena conjecture] (Gubser *et al.*, 1998; Maldacena, 1998; Witten, 1998). We also know that the BTZ black hole is related to a conformal invariant model (Horowitz and Welch, 1993; Kaloper, 1993). When the same model, with the scalar field present from the beginning, is studied numerically, though, i.e., when one is not expanding around one of these two limiting cases, which takes the scalar field vanishing, one gets a nonrational value for the same index (Burko, 2000; Garfinkle, 2001; Husain and Oliver, 2001; Pretorius and Choptuik, 2000). Since the index is calculated using numerical methods, one may not be sure if it can still be written as a rational number. At least it is certain that it is not equal to 1/2, like the numbers obtained in mean field theory calculations for critical indices. Just note that mean field theory calculations are independent of any dimensions. Mean field theory always has a Gaussian fixed point which may be considered as one of the zeroes of the beta function of the related field theory where one obtains exact conformal symmetry.

In the cases when the model can be related to the conformal symmetric theories, the hypergeometric function, the signature of conformal symmetry emerges as the solution of the problem. The presence of this function is another signature of "conformal invariance," just as "rational indices." We see the same function whether we study fluctuations in the background field of an instanton for the Yang–Mills theory (t'Hooft, 1976), or try to solve the Seiberg and Witten (1994) relations using differential equations (Bilal, 1996; Flohr, 1998). In the former case, if we do not bring in r dependent regulators, as t'Hooft does (t'Hooft, 1976), but insert only mass terms, as it is usually done, one breaks the symmetry and loses the hypergeometric functions. We may also encounter the hypergeometric functions while solving critical indices for BTZ solution (Bãnados *et al.*, 1992,1993) or for statistical models, not mentioning finding their hyperforms when we have more than one relevant variable (Appell and Kampé de Fériet, 1926; Humbert, 1920–21).

As a last example we give the emergence of this function if one expands around the exact solution for a metric in three dimensions. If one starts from a three-dimensional metric of the form (Pretorius and Choptuik, 2000).

$$ds^2 = \frac{e^{2A(r)}}{\cos^2(r)}(dr^2 - dt^2) + \tan^2(r)\, e^{2B(r)}\, d\theta^2, \tag{4}$$

which describes an AdS or BTZ solution depending on the solution one uses for the functions A and B. An exact solution, describing the AdS solution is given by (Birkandan and Hortaçsu, 2001; Pretorius and Choptuik, 2000)

$$A = -\log(\sin(r)), \tag{5}$$

$$B = \log\left(\frac{\cos(2r)}{2\sin^2(r)}\right). \tag{6}$$

If one expands the functions A and B in a power series

$$A = A^0 + \epsilon A^1 + \cdots, \tag{7}$$

$$B = B^0 + \epsilon B^1 + \cdots, \tag{8}$$

using the zeroth-order solutions as the one given above, one finds

$$A^1_{,rr} - \frac{8A^1}{\sin^2(2r)} = 0 \tag{9}$$

which can be reduced to an equation of the hypergeometric type (Birkandan and Hortaçsu, 2001). A simple calculation shows that the solutions can be written in terms of hypergeometric functions,

$$\sin^{-1}(2r) \times {}_2F_1\big(-1/2, -1/2| -1/2| \sin^2(2r)\big) \tag{10}$$

for one solution, and

$$\sin^2(2r) \times {}_2F_1\big(1, 1|5/2| \sin^2(2r)\big) \tag{11}$$

for the other. In this example, too, we see the mark of conformal symmetry, the hypergeometric functions.

In this note we gave examples of models exhibiting conformal invariance. If one signature of this symmetry is rational critical indices, the other one is the emergence of the hypergeometric functions, simple or hyper ones, in the solutions. As shown in literature (Pretorius and Choptuik, 2000; t'Hooft, 1976), when the symmetry is broken explicitly, the *rational indices* or the *hypergeometric solutions* are not present anymore. It is an open question, whether one can still construct *nontrivial models* within Lagrangian QFT or this can be done solely using the paraphernalia of *Algebraic Quantum Field Theory*.

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